TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 256, December 1979

$Z_{(2)}$-KNOT COBORDISM IN CODIMENSION TWO, AND INVOLUTIONS ON HOMOTOPY SPHERES[1]

BY

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ABSTRACT. Let $Z_{(2)}$ denote the ring of rational 2-adic integers. In this paper, we consider the group Ψ_k of $Z_{(2)}$-cobordism classes of $Z_{(2)}$-knot (Σ^{k+2}, K^k), where Σ is a 1-connected $Z_{(2)}$-sphere $Z_{(2)}$-cobordant to S^{k+2}, and K is a 1-connected $Z_{(2)}$-sphere embedded in Σ with trivial normal bundle. For $n > 3$, we will prove that $\Psi_{2n} = 0$ and $\Psi_{2n-1} = C_\varepsilon(Z_{(2)})$, $\varepsilon = (-1)^n$. Also, we will show that the group Θ_{4m-1}^{4m+1} of L-equivalence classes of differentiable involutions on $(4m + 1)$-homotopy spheres with codimension two fixed point sets defined by Bredon contains infinitely many copies of Z.

Let T be an orientation preserving differentiable involution on a homotopy sphere Σ^{k+2} with codimension two fixed point set $F(T) = K$. Then it follows from Smith theory that K is a k-dimensional $Z_{(2)}$-sphere (a manifold having the same $Z_{(2)}$ homology as S^k), where $Z_{(2)}$ denotes the ring of rational 2-adic integers. We call the pair (Σ^{k+2}, K^k) a knot.

Two such involutions T_0 and T_1 are called L-equivalent if there exists an involution T' on $\Sigma^{k+2} \times [0, 1]$ with $F(T') = M^{k+1}$, a $Z_{(2)}$-cobordism between $F(T_0)$ and $-F(T_1)$. The set of L-equivalence classes of involutions on homotopy spheres forms an abelian group Θ_k^{k+2} [4, pp. 339–340]. Ignoring the involutions, we call the two knots $(\Sigma, F(T_0))$ and $(\Sigma, F(T_1))$ L-equivalent. The set of L-equivalence classes also forms an abelian group $\theta_{k+2,k}^{(2)}$. Since the integral homology of K can be very complicated, we were unable to apply the methods in [6], [12] or [16] to compute $\theta_{k+2,k}^{(2)}$ (compare Lemma 4.2 below).

Let Σ^{k+2} denote a $Z_{(2)}$-sphere which is $Z_{(2)}$-cobordant to the standard sphere S^{k+2}, and K^k a $Z_{(2)}$-sphere [1]. Throughout this paper, we will assume that both Σ and K are 1-connected for technical reasons (see (2.7) below). An embedding of K with trivial normal bundle in Σ is called a $Z_{(2)}$-knot, denoted by (Σ^{k+2}, K^k). Two $Z_{(2)}$-knots (Σ_1, K_1) and (Σ_2, K_2) are called $Z_{(2)}$-knot corbordant (or just $Z_{(2)}$-cobordant) if there exists (M^{k+3}, N^{k+1}) with $\partial(M, N) = (\Sigma_1, K_1) \cup -(\Sigma_2, K_2)$ such that both M and N are 1-connected, the normal bundle of N in M is trivial, M is a $Z_{(2)}$-cobordism between Σ_1 and

Received by the editors December 14, 1977.

AMS (MOS) subject classifications (1970). Primary 57C45, 57D90, 57E25, 57E30.

Key words and phrases. $Z_{(2)}$-knot, $Z_{(2)}$-cobordism, simple $Z_{(2)}$-knot, Seifert manifold, Seifert matrix, normal map, normal cobordism, surgery with coefficient, involutions.

[1]Research supported by the University of Kansas General Research Fund.

$-\Sigma_2$, and N is a $Z_{(2)}$-cobordism between K_1 and $-K_2$. Let Ψ_k denote the abelian group of $Z_{(2)}$-cobordant classes of $Z_{(2)}$-knots (Σ^{k+2}, K^k). Following [16], we let $C_\varepsilon(Z_{(2)})$ (where $\varepsilon = \pm 1$) denote the group of the cobordism classes of matrices over $Z_{(2)}$ (also called the cobordism classes of ε-symmetric structures over $Z_{(2)}$ in [12], [20]). The main result of the paper is the following theorem (see (2.5) and (3.4)).

THEOREM. *For $n \geqslant 3$, there exists an isomorphism ρ_n: $\Psi_{2n-1} \to C_\varepsilon(Z_{(2)})$, $\varepsilon = (-1)^n$. Also, $\Psi_{2n} = 0$.*

Later, we will see that every element of Ψ_{2n-1} has a representative of the form (S^{2n+1}, K^{2n-1}) such that K is $(n-2)$-connected and bounds an $(n-1)$-connected $2n$-manifold in S^{2n+1}. Following [7] and [10], we will call it a simple knot. In the last section, we will study the correlation between involutions and the Seifert matrices for simple knots, and show that the group Θ^{4m+1}_{4m-1} contains infinitely many copies of Z (see Theorem 4.3 below).

Everything mentioned here will be in the differentiable category. The p.l. case can be treated in the same way by using [3].

1. Let $\theta_k^{(2)}$ denote the group of $Z_{(2)}$-cobordism classes of k-dimensional $Z_{(2)}$-spheres. If f: $K^k \to S^k$ is a $Z_{(2)}$-homology equivalence [2, p. 3], then it induces a normal map with normal invariant contained in $[S^k, G_{(2)}/O]$ [2]. We also write f for this normal map.

According to [1, (2.1)] or [2, p. 135], every element of $\theta_k^{(2)}$ can be represented by a 1-connected $Z_{(2)}$-sphere.

Let $bP_{k+1}^{(2)}$ denote the subgroup of $\theta_k^{(2)}$ consisting of all classes represented by k-dimensional $Z_{(2)}$-spheres which bound compact manifolds admitting odd frames [1]. We have the following lemma from [1, pp. 41–42].

LEMMA 1.1. *Let K_0 represent an element of $bP_{k+1}^{(2)}$, $k \geqslant 5$. Then K_0 is $Z_{(2)}$-cobordant to a homotopy sphere K_1 if $k \neq 4n-1$, and to a $(2n-2)$-connected $Z_{(2)}$-sphere K_1 which bounds a parallelizable manifold if $k = 4n-1$.*

LEMMA 1.2. *Let K_0 and K_1 be as in Lemma 1.1. If K_0 is 1-connected, then we may choose a $[(k-1)/2]$-connected $Z_{(2)}$-cobordism Q between K_0 and K_1.*

PROOF. Let I denote the interval [0, 1], and P the $Z_{(2)}$-cobordism between K_0 and K_1 in Lemma 1.1. We apply the argument in the proof of [3, Theorem 1.3] to make P 1-connected (also see [1, (2.1)]). The normal map f: $P \to S^k \times I$ with $f^{-1}(S^k \times i) = K_i$ $(i = 0, 1)$ inducing a normal map F: $P \times I \to S^k \times I \times I$. Write $Y = S^k \times I \times I$, $X = S^k \times I \times 1$, and $X_+ = S^k \times I \times 0 \cup \partial(S^k \times I) \times I$. We note that the map $F|M_+ = M_+$: $Q \times 0 \cup \partial Q \times I \to X_+$ is a $Z_{(2)}$-homology equivalence in the sense of [2, p. 3]. Then we perform surgery rel M_+ to make the map F: $(Q \times I; Q \times 1, M_+) \to (Y; X, X_+)$ normally cobordant to G: $(N^{k+2}; M^{k+1}, M_+) \to (Y; X, X_+)$ such that $G|M$

is $[(k+1)/2]$-connected and $G|N$ is $[(k+2)/2]$-connected (see Corollary 1 in [2, p. 59]).

By applying Theorem 1 in [2, p. 82], we may make G a homology equivalence of triad over $Z_{(2)}$. We take Q to be M. Since $G|M$: $M \to S^k \times I$ is $[(k+1)/2]$-connected, Q is $[(k-1)/2]$-connected. Q.E.D.

COROLLARY 1.3. *If a simply-connected $Z_{(2)}$-sphere K^k is $Z_{(2)}$-cobordant to S^k, then K bounds a $(k+1)$-dimensional $[(k-1)/2]$-connected $Z_{(2)}$-disk.*

2. Let (Σ^{k+2}, K^k) denote a $Z_{(2)}$-knot, that is, an embedding of a 1-connected $Z_{(2)}$-sphere K with trivial normal bundle in a 1-connected $Z_{(2)}$-sphere Σ which is $Z_{(2)}$-cobordant to S^{k+2}.

Let X denote the closure of $\Sigma^{k+2} - K^k \times D^2$. Since $H^1(K) = H^1(\Sigma) = H^2(\Sigma) = 0$, the map $H^1(X) \to H^1(\partial X)$ is onto from the Mayer-Vietoris sequence. Therefore, we may extend the projection map $\partial X = K \times S^1 \to S^1$ to a map g: $X \to S^1$, and thus we have the following lemma.

LEMMA 2.1. *There exists a degree 1 normal map f from (Σ^{k+2}, K^k) to the trivial knot (S^{k+2}, S^k) such that $f^{-1}(S^k) = K$ and f is a bundle map on $K \times D^2$.*

Furthermore, by making the map g transverse to a point x in S^1, we have an oriented Seifert manifold $F^{k+1} = g^{-1}(x)$ in Σ with $\partial F = K$.

LEMMA 2.2. *The $Z_{(2)}$-sphere K^k in a $Z_{(2)}$-knot (Σ^{k+2}, K^k) represents an element in $bP^{(2)}_{k+1}$.*

PROOF. It was shown in [1, (1.1)] that Σ^{k+2} admits an odd framing (that is, its stable normal bundle is $Z_{(2)}$-parallelizable). Therefore, F also admits an odd framing. Q.E.D.

LEMMA 2.3. *For $k \geqslant 5$, a $Z_{(2)}$-knot (Σ^{k+2}, K^k) is $Z_{(2)}$-cobordant to (Σ, K_1) such that $\pi_1(\Sigma - K_1) = Z$.*

PROOF. Same proof as in [14]–by making the Seifert manifold F 1-connected. Also, we note that K_1 is diffeomorphic to K. Q.E.D

As in [6], we see that f: $\Sigma - K_1 \to S^{k+2} - S^k$ is not a homology equivalence over $Z_{(2)}[Z]$ [2, p. 3], but rather a homology equivalence over $Z_{(2)}[e] = Z_{(2)}$. Using the notation of [6], we consider the surgery obstruction group $\Gamma_{k+2}(Z[Z] \to Z_{(2)}[e])$, where the map $Z[Z] \to Z_{(2)}[e]$ is the composite $Z[Z] \to Z_{(2)}[Z] \to Z_{(2)}[e]$. We may set up the surgery problem and construct the surgery group from the surgery group $L_m(Z, Z_{(2)})$ considered by Anderson [2] ($\Gamma(Z[Z] \to Z_{(2)}[Z])$ in the notation of [6]) and the map $Z_{(2)}(Z) \to Z_{(2)}[e]$, in the same way as the Cappell-Shaneson's surgery group $\Gamma_m(Z[Z] \to Z[e])$ is related to the Wall group $L_m(Z)$ [22] and the map $Z[Z] \to Z[e]$.

We call a $Z_{(2)}$-knot (Σ^{k+2}, K^k) simple, if K is $([k/2]$-1)-connected, $\pi_j(\Sigma - K) = \pi_j(S^1)$ for $j \leqslant [k/2]$, and Σ is $[k/2]$-connected.

PROPOSITION 2.4. *For $k \geqslant 5$, a $Z_{(2)}$-knot is $Z_{(2)}$-cobordant to a simple one.*

PROOF. Let (Σ^{k+2}, K^k) be a $Z_{(2)}$-knot satisfying the condition in (2.3), that is, $\Sigma - K$ is 1-connected. We construct a normal map f: $(\Sigma^{k+2}, K^k) \to (S^{k+2}, S^k)$ as in the first paragraph of the section. Define F: $f \times \text{id}$: $(\Sigma^{k+2}, K^k) \times I \to (S^{k+2}, S^k) \times I$. Let g: $Q \to S^k \times I$ be the $Z_{(2)}$-cobordism constructed in Lemma 1.2 between K and K_1 such that $g|K = f|K$. By the cobordism extension theorem, we may extend g to $Q \times D^2$ and glue it to F on $\Sigma \times I$ along $K \times D^2 \times 1$ to get a normal map G: $V \to S^{k+2} \times I$, with $Q \subseteq V$, $G^{-1}(S^k \times I) = Q$, $G|Q = g$, and G is a bundle map on a neighborhood of Q. We note that G is a normal cobordism from f: $(\Sigma, K) \to (S^{k+2}, S^k) \times 0$ to h: $(P, K_1) \to (S^{k+2}, S^k) \times 1$.

Let $E = D^{k+1} \times S^1$ denote the closure of $S^{k+2} - S^k \times D^2$. We write Y for $E \times I$, X_+ for $E \times 1$, and X for the closure of $\partial Y - X_+$. Considering the induced map $H = G|(N^{k+3}; M^{k+2}, M_+^{k+2}) \to (Y; X, X_+)$, where $N = G^{-1}(Y)$, $M = G^{-1}(X)$, and $M_+ = G^{-1}(X_+)$. We note that $P = M \cup K_1 \times D^2$ and $V = Q \times D^2 \cup N$. The map $G|M_+$ induces an isomorphism on π_1, and is a $Z_{(2)}[e]$-homology equivalence. By performing surgery rel M_+, we may assume that $H|M$ is $[(k+2)/2]$-connected, and $H|N$ is $[(k+3)/2]$-connected. Then we perform the relative surgery as in [**2**, p. 82] to make H: $(N, M, M_+) \to (Y; X, X_+)$, a homology equivalence of triads over $Z_{(2)}[e]$ (here we use the surgery with coefficient associated with $Z[Z] \to Z_{(2)}[Z] \to Z_{(2)}[e]$). By gluing back $G|Q \times D^2$ to H, we have a $Z_{(2)}$-cobordism between (Σ, K) and a $Z_{(2)}$-knot (Σ_1, K_1). If k is odd, then (Σ_1, K_1) is a simple $Z_{(2)}$-knot.

If $k = 2n$, we may take K_1 to be a homotopy sphere ((1.1) and (1.2)), and the above proof shows that $H|M \to S^1 \times D^{2n+1}$ is $(n+1)$-connected. Hence G: $\Sigma_1 \cup S^{2n} \times D^2 \to S^{2n+2}$ is $(n+1)$-connected. Therefore Σ_1 is n-connected, and $H^{n+1}(\Sigma_1) = H_{n+1}(\Sigma_1)$ is free abelian. But G is a $Z_{(2)}$-homology equivalence. Thus G is a homotopy sphere. The argument in [**11**] shows that (Σ_1, K_1) is knot cobordant to a knot with complement having the homotopy type of a circle. Q.E.D.

The last sentence in the above proof gives us the following.

COROLLARY 2.5. *For $n \geqslant 3$, $\Psi_{2n} = 0$.*

PROPOSITION 2.6. *If a $Z_{(2)}$-knot (Σ^{k+2}, K^k) is $Z_{(2)}$-cobordant to the trivial knot (S^{k+2}, S^k), $k \geqslant 5$, then (Σ, K) bounds a pair of $Z_{(2)}$-disk (W^{k+3}, B^{k+1}) such that B is $[(k-1)/2]$-connected and W is $[(k+1)/2]$-connected.*

PROOF. The proof is very similar to the ones given in (1.2) and (2.4). Hence we only give an outline here.

Let (N, Q) be a $Z_{(2)}$-cobordism between (S^{k+2}, S^k) and (Σ, K). Recall that both N and Q are 1-connected. By applying the same argument in (2.3) rel ∂N, we may require that $\pi_1(N - Q) = \pi_1(S^1)$. Furthermore, both $\pi_1(S^{k+2} - S^k) \to \pi_1(N - Q)$ and $\pi_1(\Sigma - K) \to \pi_1(N - Q)$ are isomorphisms. We then apply the argument in (1.2) rel ∂Q to construct a $Z_{(2)}$-cobordism R between Q and Q_1 such that $\partial R = \partial Q \times I$ and Q_1 is $[(k - 1)/2]$-connected. Applying the cobordism extension theorem as in (2.4) rel ∂Q, we have a normal cobordism (P^{k+4}, R^{k+2}) between (N, Q) and (N_1, Q_1) rel $\partial(N, Q) \times I$. As in (2.4), we make $\pi_j(N - Q) = \pi_j(S^1)$ for $j \leqslant [(k + 1)/2]$ and $\pi_j(P - R) = \pi_j(S^1)$ for $j \leqslant [(k + 2)/2]$. Then we make $\overline{P - R \times D^2}$ and $\overline{N_1 - Q_1 \times D^2}$ $Z_{(2)}$-cobordisms by using [2, p. 82] in the interiors as in (2.4) (as in (2.4), $\overline{N_1 - Q_1 \times D^2}$ is only homology equivalent to S^1 over $Z_{(2)}[e]$, but not over $Z_{(2)}[Z]$). Gluing $Q_1 \times D^2$ back to $N_1 - Q_1 \times D^2$, we obtain a $Z_{(2)}$-cobordism (N_1, Q_1) between (S^{k+2}, S^k) and (Σ, K) such that Q_1 is $[(k - 1)/2]$-connected and N is $[(k + 1)/2]$-connected.

Finally, we construct (W, B) by gluing the standard disks pair (D^{k+3}, D^{k+1}) to (N, Q) along (S^{k+2}, S^k). Q.E.D.

REMARK 2.7. The restriction of both Σ and K being 1-connected is used in producing a normal map f: $(\Sigma^{k+2}, K^k) \to (S^{k+2}, S^k)$ in (2.1) and a Seifert manifold F for K (Lemma 2.2). Furthermore, unlike [6], the definition of a homology equivalence f: $M \to X$ in [2, p. 3] requires that $f_{\#}$: $\pi_1(M) \to \pi_1(X)$ is an isomorphism. This condition is contained in the statement of Theorem 1 in [2, p. 82], which was used in the proof of our (1.2) and (2.4) above. If that theorem holds true without the restriction on π_1, then the proof of (1.2) would show that any Z_P-sphere K^k is Z_P-cobordant to a $([k/2] - 1)$-connected one, where P is a set of primes (compare [2, p. 135]).

3. Let $C_\varepsilon(Z_{(2)})$, $\varepsilon = \pm 1$, denote the group of cobordism classes of matrices over $Z_{(2)}$ as defined in [20, §1] (also see [12], [16]).

We first define a map ρ: $\Psi_{2n-1} \to C_\varepsilon(Z_{(2)})$, $\varepsilon = (-1)^n$, modelled on [12] and [16]. Given an element γ of Ψ_{2n-1}, we may choose a simple $Z_{(2)}$-knot $(\Sigma^{2n+1}, K^{2n-1})$ representing γ by (2.4), that is, K is $(n - 2)$-connected, Σ is $(n - 1)$-connected, and $\pi_j(\Sigma - K) = \pi_j(S^1)$ for $j \leqslant n - 1$. Let F^{2n} be a Seifert manifold. We use the argument in [14] to make F $(n - 1)$-connected. Let $H = H_n(F^{2n})$, and denote by $A(x, y)$ the Seifert linking form on H defined by computing the linking number of the cycle x with the cycle y "pushed" a small distance in the positive normal direction. Since K is just a $Z_{(2)}$-sphere, the intersection form on F: $\langle x, y\rangle = A(x, y) + \varepsilon A(y, x)$ is not unimodular, but is invertible over $Z_{(2)}$. We define $\rho(\gamma) =$ the cobordism class of A in $C_\varepsilon(Z_{(2)})$. We call A a Seifert matrix for (Σ^{2n+1}, K^{n-1}).

A $Z_{(2)}$-knot (Σ^{k+2}, K^k) is called null-cobordant if it is $Z_{(2)}$-cobordant to the trivial knot. Being null-cobordant is equivalent to the fact that (Σ^{k+2}, K^k) bounds a pair of 1-connected $Z_{(2)}$-disk (W^{k+3}, B^{k+2}).

In order to show that ρ is well defined, it suffices to prove the following lemma.

LEMMA 3.1. *If* $(\Sigma^{2n+1}, K^{2n-1})$ *is a simple null-cobordant* $Z_{(2)}$*-knot, and* A *is a Seifert matrix for* K*, then* A *is null-cobordant in* $C_\varepsilon(Z_{(2)})$.

PROOF. It follows from (2.6) that $(\Sigma^{2n+1}, K^{2n-1})$ bounds a pair of $Z_{(2)}$-disks (W^{2n+2}, B^{2n}) such that B is $(n-1)$-connected, and W is n-connected. Then the proof of Lemma 2 in **[16]** (or **[12**, p. 89], **[20**, p. 77]) can be used here. Q.E.D.

LEMMA 3.2. *For* $n \geqslant 3$, *the map* ρ *is onto.*

PROOF. Every element of $C_\varepsilon(Z_{(2)})$ can be represented by an integral matrix A with $\det(A + \varepsilon A')$ = an odd integer, where A' denotes the transpose of A. Then we construct a manifold F^{2n} with intersection form $\langle\, ,\, \rangle = A + \varepsilon A'$ by plumbing. We may take F^{2n} to be $(n-1)$-connected and $\partial F = K$ $(n-1)$-connected **[5**, Chapter V]. As in **[11**, pp. 255–257], we may embed F into S^{2n+1} and perform surgery on the complement $S^{2n+1} - F$ to realize the Seifert matrix A. We thus get a simple $Z_{(2)}$-knot (S^{2n+1}, K) with Seifert matrix A. Q.E.D.

LEMMA 3.3. *For* $n \geqslant 3$, *the map* ρ *is injective.*

PROOF. Let $(\Sigma^{2n+1}, K^{2n-1})$ be a simple $Z_{(2)}$-knot, and A its Seifert matrix associated with an $(n-1)$-connected Seifert manifold F^{2n} for K.

As in **[16**, Lemma 5], it suffices to show that a simple $Z_{(2)}$-knot $(\Sigma^{2n+1}, K^{2n-1})$ with a null-cobordant Seifert matrix A is null-cobordant. According to (1.3), Σ^{2n+1} bounds an n-connected $Z_{(2)}$-disk W^{2n+2}. Thus $(W^{2n+2}, \Sigma^{2n+1})$ is n-connected.

Since A is null-cobordant, there is a subspace G of $H_n(F)$ of one-half the rank on which A is identically zero. Hence the intersection form $A + \varepsilon A'$ is also identically zero on G. Therefore, there exist disjoint n-spheres $\{S_i\}$ embedded in F representing a basis for G. Since the corresponding linking numbers $A(x, y)$ are zero, these embeddings extend to disjoint embeddings of disks in W^{2n+2} by Haefliger's embedding theorem **[8]**. Since we may construct F alternatively by making the normal map $f\colon (\Sigma, K) \to (S^{2n+1}, S^{2n-1})$ transverse to the disk D^{2n} and then performing surgery to get $F = f^{-1}(D^{2n})$, we see that the normal bundles of $\{S_i\}$ are trivial, and the framings for the tubular neighborhoods of $\{S_i\}$ can be extended because the self-linking is zero. Then we perform surgery inside W^{2n+2} to replace F by a $Z_{(2)}$-disk (see **[12]**, **[16]**, also **[20**, (6.6)]). Q.E.D.

From the previous three lemmas, we have the following theorem:

THEOREM 3.4. *For* $n \geqslant 3$, $\rho = \Psi_{2n-1} \to C_\varepsilon(Z_{(2)})$, $\varepsilon = (-1)^n$, *is an isomorphism.*

The next corollary follows from (3.2) and (3.4).

COROLLARY 3.5. *For $n \geq 3$, every element of Ψ_{2n-1} is represented by a simple knot* [**10**, *p.* 145].

4. In this section, we will study involutions on homotopy spheres with codimension two fixed point sets [**4**, VI. 8].

According to (3.5), every element of Ψ_{2n-1}, $n \geq 3$, can be represented by a simple knot (S^{2n+1}, K^{2n-1}), where K^{2n-1} is $(n-1)$-connected, and bounds an $(n-1)$-connected manifold $F^{2n} \subseteq S^{2n+1}$. Let A be a Seifert matrix (associated with F) for K. The intersection form of F: $\langle\, , \rangle = A + \varepsilon A'$ is not unimodular, but with odd determinant. It was noted in [7, p. 52] that Levine's classification theorem for simple spherical knots also holds in this more general context–two simple knots are isotopic in S^{2n+1} if and only if their Seifert matrices are related by a chain of congruence, elementary enlargements, and elementary reductions [**17**]. We will use Seifert matrices to describe which simple knots arise as fixed point sets of involutions on homotopy spheres.

We will let Σ and Σ' denote homotopy spheres in the rest of the paper. Also we let Σ_0 denote the generator of bP_{2n+2} [**13**]. Recall that $\varepsilon = (-1)^n$.

THEOREM 4.1. *If (S^{2n+1}, K^{2n-1}), $n \geq 3$, is a simple knot and Σ^{2n+1} a homotopy sphere, then $\Sigma^{2n+1} = \Sigma^{2n+1} \# S^{2n+1}$ admits an involution T with K as its fixed point set and with orbit space $\Sigma' = \Sigma / T$ if and only if*

(a) *(S^{2n+1}, K^{2n-1}) has a Seifert matrix B of the form $B = A(A - \varepsilon A')^{-1}A$ for some integral matrix A with* $\det(A + \varepsilon A')$ *= an odd integer and* $\det(A - \varepsilon A') = \pm 1$.

(b) *For n odd, $\Sigma = s\Sigma_0 + 2\Sigma'$, where s = signature$(A - \varepsilon A')$. For n even, $\Sigma = a\Sigma_0 + 2\Sigma'$, where a = the Arf invariant of A.*

PROOF. This is just a slight modification of the proof for spherical simple knots given in [**19**].

The orbit map $\Sigma \to \Sigma'$ is a 2-fold branched covering. Let A be a Seifert matrix for the simple knot (Σ', K). Hence $\det(A + \varepsilon A')$ = an odd integer. Corollary (5.7) of [**10**] shows that Σ bounds an n-connected parallelizable manifold with intersection form $A - \varepsilon A'$. Therefore Σ is a homotopy sphere if and only if $\det(A - \varepsilon A') = \pm 1$.

The rest of the proof is almost the same as that of [**19**]. In the proof of Lemma 2 of [**19**], we multiply equation (3) by $(A + \varepsilon A')^{-1}$. Since $(A + \varepsilon A')^{-1}$ exists over the rationals Q, the same proof carries through. We refer the readers to [**19**] for details. Q.E.D.

Let Θ_k^{k+2} denote the group (under connected sum) of L-equivalence classes of involutions on homotopy spheres with codimension two fixed point sets (see [**4**, p. 340]).

Let K^{2k-1} be a $Z_{(2)}$-sphere embedded in a homotopy sphere Σ^{2n+1} with trivial normal bundle. Let X denote the closure of $\Sigma - K \times D^2$. By using the Poincaré duality and Alexander duality, we see that $H^2(X, \partial X) = H_n(X) = H^1(K) = 0$. Therefore, there exists a Seifert manifold F^{2n} for (Σ, K) [14, Lemma 2]. As in [12], [16] or [20, p. 77], we may use $H_n(F^{2n})/\text{Torsion}$ to construct a Seifert matrix, and define a map $\rho'\colon \theta^{(2)}_{2n+1,2n-1} \to C_\varepsilon(Z_{(2)})$, where $\varepsilon = (-1)^n$.

We may use the arguments in [12], [16] or [20, (6.6)] to show that ρ' is well defined and surjective (we need a highly connected Seifert manifold to prove the injectivity in [12], [16] or [20]). Thus we have the following lemma:

LEMMA 4.2. *For $n \geqslant 3$, $\rho'\colon \theta^{(2)}_{2n+1,2n-1} \to C_\varepsilon(Z_{(2)})$, $\varepsilon = (-1)^n$, is surjective.*

THEOREM 4.3. *For $m \geqslant 2$, Θ^{4m+1}_{4m-1} contains infinitely many copies of Z.*

PROOF. Because of (4.1) and (4.2), it suffices to show that there exist infinitely many linearly independent integral Seifert matrices $\{A_k\}$ in $C_{+1}(Z_{(2)})$ satisfying $\det(A_k + A_k') =$ an odd integer and $\det(A_k - A_k') = \pm 1$. (Notice here $\varepsilon = (-1)^{2m} = +1$.)

From [16, p. 243], we have the following sequence of linearly independent elements $\{A_k\}$ $(k = 1, 2, \dots)$ in $C_{+1}(Z)$:

$$A_k = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is known that an integral matrix A is null-cobordant over Z if and only if it is null-cobordant over the rationals Q [15], [16]. Therefore, $\{A_k\}$ are linearly independent over $Z_{(2)}$.

Because both $A_k + A_k'$ and $A_k - A_k'$ are unimodular, the fixed point sets of the involutions constructed are homotopy spheres [19]. Q.E.D.

Next we consider the question of whether every element of $bP^{(2)}_{4m}$ can be realized as the fixed point set of some involution on a $(4m+1)$-homotopy sphere Σ. (For $bP^{(2)}_{4m+2}$, see [18].) According to [1, p. 41] (also see (1.1)), all elements of $bP^{(2)}_{4m}$ can be realized as the $(2m-2)$-connected boundaries of $(2m-1)$-connected manifolds constructed by plumbing with respect to symmetric, even matrices with odd determinants. Over $Z_{(2)}$, these matrices can be expressed as sums of copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ [9]. The argument in [21] will give us involutions on homotopy sphere Σ^{4m+1} with elements of $bP^{(2)}_{4m}$ corresponding to the above two matrices as fixed point sets (also see [4, p. 341]). By taking connected sums, we can realize all of them.

PROPOSITION 4.4. *Every element $bP^{(2)}_{4m}$ has a representative which is the fixed point set of an involution on a homotopy sphere Σ^{4m+1}.*

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